

Instanton and Monopole in External Chromomagnetic Fields

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We study properties of instanton and monopole in an external chromomagnetic field. Generally, the 't Hooft ansatz is no longer a solution of the Yang-Mills field equation in the presence of external fields. Therefore, we investigate a stabilized instanton solution with minimal total Yang-Mills action in a nontrivial topological sector. With this aim, we consider numerical minimization of the action with respect to the global color orientation, the anisotropic scale transformation and the local gauge-like transformation starting from a simple superposed gauge field of the 't Hooft ansatz and the external color field. Here, the external color field is, for simplicity, chosen to be a constant Abelian magnetic field along a certain direction. Then, the 4-dimensional rotational symmetry $O(4)$ of the instanton solution is reduced to two 2-dimensional rotational symmetries $O(2) \times O(2)$ due to the effect of a homogeneous external field. In the space \mathbf{R}^3 at fixed t , we find a quadrupole deformation of this instanton solution. In the presence of a magnetic field \vec{H} , a prolate deformation occurs along the direction of \vec{H} . Contrastingly, in the presence of an electric field \vec{E} an oblate deformation occurs along the direction of \vec{E} . We further discuss the local correlation between the instanton and the monopole in the external field in the maximally Abelian gauge. The external field affects the appearance of the monopole trajectory around the instanton. In fact, a monopole and anti-monopole pair appears around the instanton center, and this monopole loop seems to partially screen the external field.

§1. Introduction

Quantum chromodynamics (QCD) includes various infrared phenomena, such as color confinement, dynamical chiral-symmetry breaking and large η' mass. It is necessary to clarify nonperturbative phenomena of the QCD vacuum composed of quarks, anti-quarks and gluon fields interacting in a highly complicated way. Topological properties may provide a useful approach for descriptions of the nonperturbative nature of QCD, although the usual perturbative expansion is not applicable in this infrared region. The instanton solution¹⁾ is actually related to the $U_A(1)$ anomaly and large η' mass.²⁾ Witten and Veneziano derived an approximate relation between the η' mass and topological susceptibility in the context of the $1/N_c$ expansion.^{3), 4)} Chiral symmetry breaking could also be interpreted as an instanton effect.^{5) - 7)}

With recent progress in computational capabilities, the direct investigation of instanton properties in the QCD vacuum using lattice QCD simulations has become practicable.^{8) - 15)} Elucidating instanton physics is important in order to understand the nonperturbative QCD vacuum. A cooling procedure can be adopted to eliminate short-range quantum fluctuations and extract only topological excitations.⁸⁾ Instantons seem to appear as deformed configurations even after cool-

ing. The vacuum structure of non-Abelian gauge theories has been studied by Savvidy employing the background field method.¹⁶⁾ In the $SU(2)$ Yang-Mills theory, the effective potential up to one loop order for an Abelian gauge field leads to a nonzero constant color magnetic field instead of the perturbative vacuum. Following Savvidy's work, the Copenhagen group described the QCD vacuum including the effect of the inhomogeneous magnetic structure.^{17) - 19)} Lattice QCD simulations yield an average instanton size $\bar{\rho} \simeq (0.33 - 0.4)$ fm and an instanton number density $(N/V) \sim \langle G_{\mu\nu} G_{\mu\nu} / 32\pi^2 \rangle \simeq 1 \text{ fm}^{-4}$, corresponding to the gluon condensate.^{8) - 15), 20)} The QCD vacuum contains the gluon condensate and the inhomogeneous magnetic field. The 't Hooft ansatz is not a solution of the Yang-Mills equation in the presence of such a background field, and therefore the position, size, color orientation and 4-dimensional shape of each instanton are altered by the field. In this paper, we study the instanton deformation mechanism by deriving a stabilized instanton solution that corresponds to a minimum Yang-Mills action in the external field.

Recent studies have revealed the remarkable fact that instantons have a one-to-one correspondence with monopoles in the Abelian gauge, although these topological objects belong to different homotopy groups.^{21) - 26)} This correspondence may provide a scenario of a color confinement mechanism that the instanton fields give rise to the multi-production of monopole loops as a signal of monopole condensation.^{24), 25)} For this reason, we further investigate the appearance of monopoles around the instanton as a deformed configuration in external color magnetic fields.

§2. Instantons in an external field

The instanton is a classical nontrivial solution of the Euclidean Yang-Mills field equations with a finite action, which was discovered by Belavin, Polyakov, Schwartz and Tyupkin.^{1), 27), 28)} The appearance of instantons corresponds to the homotopy group $\Pi_3(SU(N_c)) = Z_\infty$.²⁷⁾ In Minkowski space, instantons are interpreted as tunneling events among degenerate vacua that are labeled by different winding numbers. The Euclidean Yang-Mills action is written

$$S = \frac{1}{2g^2} \int d^4x \text{tr} (G_{\mu\nu} G_{\mu\nu}), \quad (2.1)$$

where $G_{\mu\nu} \equiv [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$ is the field strength.²⁷⁾ Here, to simplify the notation, the field strength and the gauge field are defined as $G_{\mu\nu} \equiv iG_{\mu\nu}^a \tau^a / 2$ and $A_\mu \equiv iA_\mu^a \tau^a / 2$. By defining the dual field strength as $\tilde{G}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} G_{\alpha\beta}$, we can rewrite the action as

$$S = \frac{1}{8g^2} \int d^4x \{ G_{\mu\nu}^a \pm \tilde{G}_{\mu\nu}^a \}^2 \mp \frac{8\pi^2}{g^2} Q. \quad (2.2)$$

From this formula, the action provides a minimal value characterized by the topological charge $Q \equiv (1/32\pi^2) \int d^4x G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a$, when the field strength satisfies the (anti)self-dual condition $G_{\mu\nu}^a = \pm \tilde{G}_{\mu\nu}^a$. This self-dual relation satisfies the Yang-Mills field equation automatically, as $[D_\mu, G_{\mu\nu}] = \pm [D_\mu, \tilde{G}_{\mu\nu}] = 0$, due to the Jacobi identity.

The instanton solution is well known as the 't Hooft ansatz.^{27), 28)} The single instanton solution with $\mathcal{Q} = 1$ is written

$$A_\mu^{I(s)}(x) = \frac{2iO^{ab}\bar{\eta}^{b\mu\nu}(x-z)_\nu\rho^2}{(x-z)^2\{(x-z)^2+\rho^2\}} \frac{\tau^a}{2} \quad (2.3)$$

in the singular gauge. The $SU(2)$ instanton has 8 collective coordinates corresponding to the size ρ , the 4-dim central position z_μ , and the global orientation O^{ab} in color space. Here, $\eta_{b\mu\nu}$ denotes the 't Hooft symbol,²⁷⁾ defined as

$$\bar{\eta}^{b\mu\nu} = -\bar{\eta}^{b\nu\mu} \equiv \begin{cases} \varepsilon^{b\mu\nu}, & (\mu, \nu = 1, 2, 3) \\ -\delta^{b\mu}, & (\nu = 4) \end{cases} \quad (2.4)$$

with $\eta_{b\mu\nu} \equiv (-1)^{\delta^{\mu 4} + \delta^{\nu 4}} \bar{\eta}_{b\mu\nu}$. The anti-instanton with $\mathcal{Q} = -1$ can be obtained by replacing $\bar{\eta}_{a\mu\nu}$ with $\eta_{a\mu\nu}$ in Eq. (2.3). The single instanton action density has a point-like peak at its center with the 4-dimensional rotational symmetry $O(4)$ in the space-time \mathbf{R}^4 .

The QCD vacuum contains, for instance, instanton and anti-instanton fields, the gluon condensate and inhomogeneous magnetic fields, as in the Copenhagen vacuum, and so on. Therefore, we would like to study the effects of background fields on classical configurations like instantons. For simplicity, it is convenient to consider a translationally invariant external field. In the $SU(2)$ case, there are two categories of such external fields. By employing a suitable gauge, one is reduced to a QED-like case, and the other is reduced to a constant A_μ^1 and A_ν^2 case, which lead to a constant field strength $F_{\mu\nu}^3 \propto [A_\mu^1, A_\nu^2]$. In this work, we consider the QED-like case as an idealized external field. We use an external gauge field given by

$$A_\mu^{3ex} = -\frac{1}{2}F_{\mu\nu}x_\nu, \quad A_\mu^{\pm ex} \equiv \frac{1}{\sqrt{2}}(A_\mu^1 \pm iA_\mu^2) = 0, \quad (2.5)$$

which leads to a constant Abelian field strength $G_{\mu\nu} = iF_{\mu\nu}\frac{\tau^3}{2}$. Here, $F_{\mu\nu} \in \mathbf{R}$ is a constant anti-symmetric tensor.

In the presence of an external field, the 't Hooft ansatz is no longer a solution of the Yang-Mills field equation. Therefore, we would like to find the instanton solution that minimizes the total action in external fields. First of all, we construct the total gauge field as a superposition of the instanton field and the external field:

$$A_\mu^{tot}(x) = A_\mu^{ex}(x) + A_\mu^I(x). \quad (2.6)$$

The total Yang-Mills action given by this simple superposed configuration would be shifted from the absolute minimum due to the overlapping between A_μ^{ex} and A_μ^I . Therefore, we derive a stabilized solution with nontrivial topological sector that minimizes the total Yang-Mills action by employing a variational analysis with respect to the instanton configuration A_μ^I .

§3. Minimization of total instanton action under an external field

As the degrees of freedom of the variational space, we consider (i) the global color orientation O^{ab} , (ii) the anisotropic scale transformation λ_μ , and (iii) the local

gauge-like transformation $\Omega^a(x)$ of the instanton configuration. Certainly, the action of an isolated single instanton is independent of O and $\Omega(x)$ due to the gauge invariance. However, the total action depends directly on these variables in the presence of external fields. We should start from the global variables and then perform the minimization locally in order to find the position of absolute minimum in this variational space. In the actual calculation, we determine O^{ab} , λ_μ and $\Omega(x)$ in turn by minimizing the total Yang-Mills action with respect to each.

3.1. Global color orientation

First, we minimize the total action with respect to the global color orientation O^{ab} . A single instanton has collective coordinates concerning the global rotation in the $SU(2)$ color space given by

$$O \equiv \begin{bmatrix} -\sin \psi \sin \phi + \cos \psi \cos \theta \cos \phi & \sin \psi \cos \phi + \cos \psi \cos \theta \sin \phi & -\cos \psi \sin \theta \\ -\cos \psi \sin \phi - \sin \psi \cos \theta \cos \phi & \cos \psi \cos \phi - \sin \psi \cos \theta \sin \phi & \sin \psi \sin \theta \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \end{bmatrix}, \quad (3.1)$$

which is characterized by the three Euler angles $\theta \in [0, \pi]$ and $\phi, \psi \in [0, 2\pi)$. In the absence of external fields, the global $O(4)$ symmetry is manifest, and these variables are irrelevant to the action. However, if there is an external field breaking the 4-dimensional space-time symmetry, these Euler angles would be determined from the condition of a minimal total action.

3.2. Anisotropic scale transformation

As the next step, we carry out an anisotropic scale transformation of the instanton configuration as

$$x_\mu \rightarrow \lambda_\mu x_\mu. \quad (3.2)$$

Here, λ_μ are the global rescale parameters acting on the coordinates x_μ along the direction μ . This anisotropic scale transformation directly affects the instanton profile, which is not done by a transformation of the gauge degrees of freedom of the instanton configuration. Here, we impose the condition $\lambda_1 \lambda_2 \lambda_3 \lambda_4 = 1$ and fix the 4-dimensional volume of the instanton profile in terms of the characteristic scale $\bar{\rho}$. At this point, we should give a comment on the size of an instanton. Generally, the overlapping between an instanton and the background field increases the total action density. A larger size of the instantons leads to a much larger increase of the action, and the stability of a finite size instanton is not certain in a uniform external field. However, in the QCD vacuum, instantons and anti-instantons of a characteristic size and density have been observed using lattice QCD simulations.^{8) - 15), 20)} It has been found also in the simplified instanton gas model that the instanton size distribution has a peak at a characteristic instanton size.²⁹⁾ With these results in mind, we here impose the above condition by hand and consider only the deformation effect on an instanton in such a characteristic size. The breaking of the $O(4)$ symmetry due to external fields leads to a nontrivial condition of λ_μ , which minimize the total action.

3.3. Local gauge-like transformation

Until this point, we have considered only the global parameters of the instanton configuration. The self-duality of an instanton solution is generally broken in the presence of an external field, and the true minimum of the total action cannot be realized through a global transformation alone for such a self-dual initial configuration. For this reason, we also consider the local minimization of the total action considering gauge-like degrees of freedom of the instanton configuration. We formulate the gauge-like transformation as follows. In the construction of the total gauge fields, there is an ambiguity of the “gauge-like choice” of $A_\mu^I(x)$ as

$$A_\mu^{I'}(x) = \Omega(x)(A_\mu^I(x) + \partial_\mu)\Omega^\dagger(x), \quad (3.3)$$

in the presence of an external $A_\mu^{ex}(x)$, whose gauge degrees of freedom are fixed as

$$A_\mu^{ex'}(x) = A_\mu^{ex}(x). \quad (3.4)$$

For instance, the instanton gauge field $A_\mu^{I\text{sing}}$ in the singular gauge can be expressed equivalently as

$$A_\mu^{I\text{reg}} = \Omega^{I\text{sing}}(A_\mu^{I\text{sing}} + \partial_\mu)\Omega^{\dagger I\text{sing}}, \quad (3.5)$$

in the regular gauge with the singular gauge function $\Omega^{I\text{sing}} = (x_4 + x_i\tau_i)/|x|$. Certainly, the gauge invariant quantities, like the instanton action S^I and topological charge \mathcal{Q}^I , are independent of this gauge choice in the absence of external fields. However, external fields cause the total action S^{tot} to depend on the “gauge-like choice” of the instanton part A_μ^I , like Eq. (3.5), although S^{tot} is, of course, independent of the “total” gauge transformation of $A_\mu^{\text{tot}} = A_\mu^{ex} + A_\mu^I$ given by

$$A_\mu^{\text{tot}'}(x) = \Omega(x)(A_\mu^{\text{tot}}(x) + \partial_\mu)\Omega^\dagger(x). \quad (3.6)$$

We call this partial gauge transformation of Eqs. (3.3) and (3.4) a “gauge-like” transformation. Such a “gauge-like” function should be determined iteratively by the local minimization of the total action S^{tot} .

§4. Lattice formulation

The lattice technique is applicable to the local minimization of the total Yang-Mills action for the gauge-like function $\Omega(x)$. Using the lattice, we can additionally employ the maximally Abelian (MA) gauge fixing and extract monopole currents as will be considered in §6. The link variables both for the instanton and the external field can be defined as

$$U_\mu^I(s) \equiv \exp[iaA_\mu^I(s)], \quad (4.1)$$

$$U_\mu^{ex}(s) \equiv \exp[iaA_\mu^{ex}(s)], \quad (4.2)$$

respectively. We use the standard Wilson action as the lattice Yang-Mills action, which can be rewritten as

$$S_{YM}^{\text{tot}} = \frac{1}{8\pi^2} \sum_{s, |\mu| > |\nu|, (ij)} \text{tr} \left[1 - P_{\mu\nu}^{(ij)}(s) \right] + \text{h.c.}, \quad (4.3)$$

with the plaquette variable

$$P_{\mu\nu}^{(ij)} \equiv U_{\mu}^i(s) U_{\nu}^j(s + \hat{\mu}) U_{\mu}^{i\dagger}(s + \hat{\nu}) U_{\nu}^{j\dagger}(s). \quad (4.4)$$

Here, the summation over μ and ν is taken as $\mu, \nu \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$. With regard to the preservation of 4-dim geometrical symmetry, this clover-type action seems preferable. The labels $i, j \in \{R, L\}$ relate to the ordering ambiguity in the definition of the total link variables due to the lattice discretization. In fact, we have the two choices

$$U_{\mu}^R(s) \equiv U_{\mu}^{ex}(s) U_{\mu}^I(s), \quad (4.5)$$

$$U_{\mu}^L(s) \equiv U_{\mu}^I(s) U_{\mu}^{ex}(s), \quad (4.6)$$

which reproduce $A_{\mu}^{tot}(x) = A_{\mu}^{ex}(x) + A_{\mu}^I(x)$ in the continuum limit. We actually combine U_{μ}^R and U_{μ}^L in such a manner to maintain the geometrical symmetry and the ordering symmetry for the product of U_{μ}^I and U_{μ}^{ex} , as shown in the Appendix.

Under the anisotropic scale transformation following Eq. (3.2), the instanton link variables become

$$U_{\mu}^I(\lambda_{\alpha} x_{\alpha}) \equiv \exp \left(i \lambda_{\mu} a A_{\mu}^I(\lambda_{\alpha} x_{\alpha}) \right). \quad (4.7)$$

Here, it is noted that the integral elements a are replaced with $\lambda_{\mu} a$.

The local gauge-like transformation of Eqs. (3.3) and (3.4) at a site s_0 is defined on the lattice as

$$U_{\mu}^{I'}(s_0) = \Omega(s_0) U_{\mu}^I(s_0) \Omega^{\dagger}(s_0 + \hat{\mu}), \quad (4.8)$$

$$U_{\mu}^{ex'}(s_0) = U_{\mu}^{ex}(s_0), \quad (4.9)$$

with $\Omega(s_0) = \Omega_0(s_0) \mathbf{1} + i \Omega_i(s_0) \tau_i$, while $U_{\mu}^I(s)$ and $U_{\mu}^{ex}(s)$ are fixed at the sites $s \neq s_0$. Under the local gauge-like transformation $\Omega(s_0)$, the Wilson action density at s_0 changes as $S_{YM}(s_0) \rightarrow S'_{YM}(s_0)$, with $S'_{YM}(s_0) = S_{YM}(s_0) - \delta S_{YM}(s_0; \Omega(s_0))$. Apart from an irrelevant constant, the difference of the total action density, δS_{YM} , can be written in a bilinear form of Ω ,

$$\delta S_{YM}(s_0; \Omega(s_0)) = \sum_{\mu > \nu} \sum_{\alpha=1}^{12} \text{tr} \left[\hat{L}_{\mu\nu}^{\alpha} \Omega \bar{L}_{\mu\nu}^{\alpha} \Omega^{\dagger}(s_0) \right], \quad (4.10)$$

where $\hat{L}_{\mu\nu}^{\alpha}$ and $\bar{L}_{\mu\nu}^{\alpha}$ are given by U^I and U^{ex} as shown in the Appendix. At the boundary of the N^4 lattice, we employ a trivial boundary as $\Omega^0(s) = 1$ and $\Omega^i(s) = 0$. We can determine the appropriate function $\Omega(s)$ by maximizing δS at each site, and we obtain the instanton solution with minimal total action iteratively.

§5. Numerical results

In the actual calculation, we considered a typical size instanton with $\rho = 0.4 \text{ fm}$.²⁸⁾ Although a single instanton possesses scale invariance in the continuum space, a

small-size instanton with $\rho < a$ cannot be described on a lattice because of the finite lattice spacing a . In order to guarantee lattice continuity for instanton configurations, we used a lattice spacing $a = 0.05$ fm that is quite fine compared to the instanton size $\rho = 0.4$ fm on a $N^4 = 32^4$ lattice. From the instanton liquid model,²⁸⁾ we have another scale parameter, the gluon condensate, $\frac{1}{32\pi^2} \langle G_{\mu\nu}^a G_{\mu\nu}^a \rangle \simeq (200 \text{ MeV})^4$. As a typical magnitude of external fields, $F_{\mu\nu}$ is set to be on the order of the gluon condensate. Without loss of generality, we take $F_{12} > 0$ and set the other components equal to zero in a suitable Lorentz frame. This case corresponds to that of a constant magnetic system.

We adopt the 't Hooft ansatz in the singular gauge as the initial instanton configuration superposed on the external field. Although we can also use the regular gauge for the instanton, the regular gauge function spreading over a wide region leads to a larger increase in the action due to the overlapping with the external field than in the case of the singular gauge. This is the practical reason why we use the singular gauge initially. Certainly, we would obtain the same result starting from different gauge choices.

Figure 1 shows the dependence of the total action on the global color orientation of Eq. (3.1). Here, we plot contribution from the cross term between the instanton and external field to the total action, defined as

$$S^{\text{cross}}/S^I \equiv (S^{\text{tot}} - S^I - S^{\text{ex}})/S^I, \quad (5.1)$$

which is normalized by the isolated instanton action S^I . Here, S^{tot} and S^{ex} denote the total Yang-Mills action and the external action, respectively. This variable S^{cross} differs from S^{tot} only by a constant, since S^I and S^{ex} are constants.

The total action is independent both of the angles ψ and ϕ in Eq. (3.1), because of the $O(2)$ symmetry along the direction of the external field. In sharp contrast, the total action depends rather strongly on θ , as shown in Fig. 1. We find the value $\theta = 0$ that minimizes the total action for the global color orientation. In the absence of external fields, these parameters of the single instanton are collective coordinates, and cannot be determined uniquely.

After determining the global angles, we carried out the anisotropic transformation of the instanton profile given by Eq. (4.7). It is instructive to consider the case without an external field, in which the instanton is a classical solution of the Yang-Mills field equation. An arbitrary anisotropic scale transformation leads to an increase of the action, where the minimal action condition is trivial at $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$, as shown in Fig. 2(a). Thus, the single instanton is found to be stable with 4-dimensional spherical symmetry.

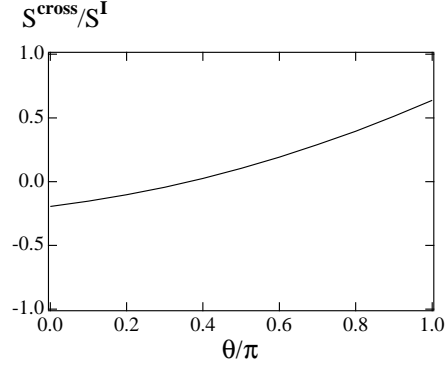


Fig. 1. The dependence of the total action on the global color-orientation angle θ in a homogeneous external field with $F_{12} > 0$.

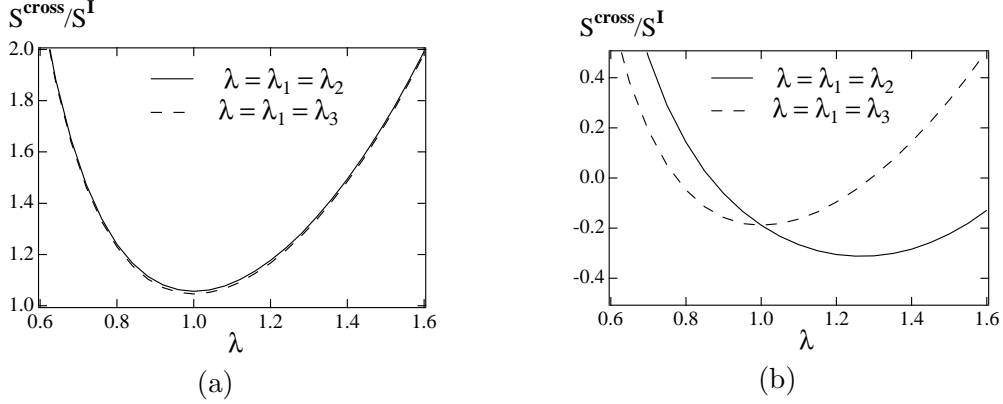


Fig. 2. The dependence of the total action on the anisotropic scale transformation with λ_μ . The solid curve represents the result with fixed $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_3 = \lambda_4 = 1/\lambda$, and the dashed curve represents that $\lambda_1 = \lambda_3 = \lambda$ and $\lambda_2 = \lambda_4 = 1/\lambda$. (a) The isolated instanton case without an external field. (b) The interacting instanton case with an external field $F_{12} > 0$.

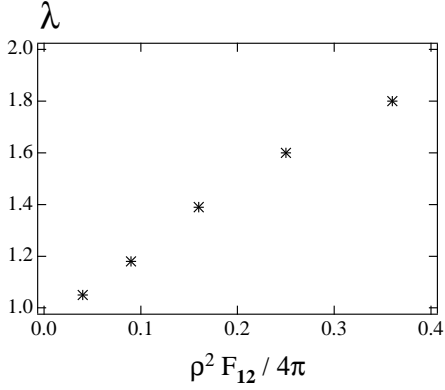


Fig. 3. The dependence of the magnitude of the instanton deformation on the size ρ and the strength of the external field F_{12} . We plot the parameter λ that provides the minimal Yang-Mills action as a function of the dimensionless variable $\rho^2 F_{12}/4\pi$.

The external field leads to a deformation of the instanton profile. Here, it is noted that the 4-dimensional rotational symmetry $O(4)$ is reduced to two 2-dimensional rotational symmetries $O(2) \times O(2)$, due to the homogeneous external field $F_{12} > 0$. The scale transformation with fixed $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$ provides a nontrivial minimum point, as shown in Fig. 2(b), although the total action tends to increase under the deformation with fixed $\lambda_1 = \lambda_3$ and $\lambda_2 = \lambda_4$. In this external field, the instanton is stable for a quadrupole-deformed configuration in \mathbf{R}^3 . From this result, we determine the parameters λ_μ as $\lambda_1 = \lambda_2 = \lambda = 1.3$ and $\lambda_3 = \lambda_4 = 1/\lambda$ to realize the minimal

total action with respect to the anisotropic transformation. Figure 3 shows that both the instanton size and the strength of the external field affect the deformation of the instanton profile. The parameter $(\lambda - 1)$ is approximately proportional to the dimensionless parameter $\rho^2 F_{12}$. It is found that the instanton would be deformed more strongly by the background fields as the size increases.

To this point, we have determined the global color orientation and the anisotropic scale transformation as global conditions. In order to obtain a stable instanton solution in the external field, we further consider the gauge-like transformation of Eqs. (3.3) and (3.4) as a local minimization of the total Yang-Mills action. We plot the cooling curve for the local gauge-like transformation in Fig. 4. The ini-

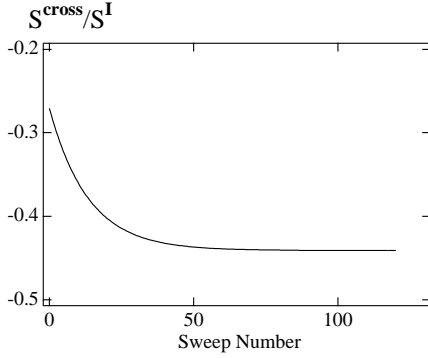


Fig. 4. The cooling curve of the total action for the local gauge-like transformation of $A_\mu^I(\theta_{min}, \lambda_{min})$. The initial configuration is the 't Hooft ansatz in the singular gauge with $\theta_{min} = 0$ and $\lambda_{min} = 1.3$.

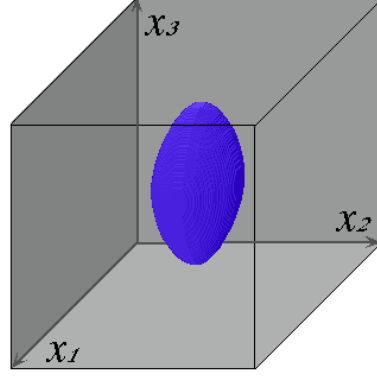


Fig. 5. The total action density of the instanton solution in an external field in a cross section of \mathbf{R}^3 in the t plane, including the instanton center. The presently considered external field has a nonzero magnetic component H_3 along the third direction, x_3 .

tial configuration in Fig. 4 was determined previously to be the 't Hooft ansatz of Eq. (2.3) in the singular gauge with the global parameter values $\theta_{min} = 0$ and $\lambda_1 = \lambda_2 = \lambda_{min} = 1.3$. With this iteration, the total action further decreases and converges after about 100 sweeps. Consequently, we obtain the stable instanton solution with minimal total Yang-Mills action, which cannot be obtained by considering only global parameters like O^{ab} and λ . Finally, we note that the instanton seems to be a quadrupole-deformed configuration in \mathbf{R}^3 , as shown in Fig. 5.

To summarize this section, we considered the instanton deformation mechanism in a homogeneous external color field. In an external field with $F_{12} > 0$, the total action depends strongly on the angle θ of the global color orientation and the parameters λ_μ of the anisotropic scale transformation. We obtained the minimal total Yang-Mills action at $\theta = 0$ and $\lambda_1 = \lambda_2 = 1/\lambda_3 = 1/\lambda_4 > \lambda = 1.3$. The magnitude of the instanton deformation depends both on the instanton size and the strength of the external field. A large instanton seems to be affected and deformed strongly by background fields. Finally, we obtained a stable instanton solution where the $O(4)$ symmetry is reduced to two 2-dimensional rotational symmetries $O(2) \times O(2)$ in the xy and zt planes, due to the homogeneous external field with $F_{12} = H_3 > 0$. In \mathbf{R}^3 at a fixed t , we find a quadrupole deformation of this instanton solution. In presence of a magnetic field \vec{H} , a prolate deformation occurs along the direction of \vec{H} . In an electric field \vec{E} , which corresponds to e.g. $F_{14} \neq 0$, an oblate deformation occurs along the direction of \vec{E} . Due to the external field, the instanton solution loses its 4-dimensional rotational symmetry.

§6. Monopole around an instanton in an external field

In 1981, 't Hooft proposed the Abelian gauge,³⁰⁾ for which the $SU(N_c)$ non-Abelian gauge theory is reduced to the $U(1)^{N_c-1}$ Abelian gauge theory with color-

magnetic monopoles. The appearance of magnetic monopoles corresponds to the homotopy group $\Pi_2(SU(N_c)/U(1)^{N_c-1}) = Z_\infty^{N_c-1}$, which is different from that of instantons. In the Abelian gauge, the color-magnetic monopole appears as a relevant degree of freedom for the description of color confinement, which was proposed by Nambu, 't Hooft and Mandelstam in the mid-1970s.³¹⁾⁻³⁴⁾ This mechanism can be interpreted as the dual Meissner effect due to monopole condensation, which is a dual version of Cooper pair condensation in ordinary superconductivity. This analogy is based on the duality between the magnetic and electric parts. In lattice QCD simulations, it has been observed that large monopole clustering covers the entire physical vacuum in the confinement phase in the MA gauge. This has been cited as evidence that monopole condensation is responsible for confinement.^{35), 36)} Many studies indicate that the monopole is a relevant degree of freedom for color confinement and chiral symmetry breaking.^{37) - 45)}

Recent studies show that monopoles are directly related to instantons, although these topological objects belong to different homotopy groups.^{21) - 26)} This correlation suggests that instantons are important for the multi-production of monopole loops.^{24), 25)} For this reason, we investigate further the background-field effect on the appearance of monopoles around instantons.

The monopole current is extracted as follows. We use the MA gauge,³⁵⁾ which is defined in the $SU(2)$ case by minimizing

$$R_{ch} = 2 \sum_{s, \mu} \left[1 - \frac{1}{2} \{ (U_\mu^1(s))^2 + (U_\mu^2(s))^2 \} \right], \quad (6.1)$$

with $U_\mu = U_\mu^0 + i\tau^i U_\mu^i$. In the MA gauge, the link variable is decomposed as $U_\mu(s) = M_\mu(s)u_\mu(s)$, with

$$M_\mu(s) \equiv \begin{pmatrix} \sqrt{1 - |c_\mu(s)|^2} & -c_\mu^*(s) \\ c_\mu(s) & \sqrt{1 - |c_\mu(s)|^2} \end{pmatrix}, \quad u_\mu(s) \equiv \begin{pmatrix} e^{i\theta_\mu(s)} & 0 \\ 0 & e^{-i\theta_\mu(s)} \end{pmatrix}, \quad (6.2)$$

where the Abelian angle variable θ_μ and the non-Abelian variable c_μ are defined in terms of U_μ as $\tan \theta_\mu = U_\mu^3/U_\mu^0$ and $c_\mu e^{i\theta_\mu} = -U_\mu^2 + iU_\mu^1$. It is obvious from the expression of Eq. (6.1) that the off-diagonal parts U_μ^1 and U_μ^2 of the gluon fields are minimized in the MA gauge. Therefore, full $SU(2)$ link variables are approximated as $U_\mu \simeq u_\mu$, in the MA gauge.

Monopole currents are defined by $u_\mu(s)$, following DeGrand and Toussaint.⁴⁶⁾ Using the forward derivative $\partial_\mu f(s) \equiv f(s + \hat{\mu}) - f(s)$ with unit vector $\hat{\mu}$, the 2-form of the lattice formulation, $\theta_{\mu\nu}(s) \equiv \partial_\mu \theta_\nu(s) - \partial_\nu \theta_\mu(s)$, is decomposed as

$$\theta_{\mu\nu}(s) = \bar{\theta}_{\mu\nu}(s) + 2\pi n_{\mu\nu}(s), \quad (6.3)$$

with $\bar{\theta}_{\mu\nu}(s) \equiv \text{mod}_{2\pi} \theta_{\mu\nu} \in (-\pi, \pi]$ and $n_{\mu\nu}(s) \in \mathbf{Z}$. Here, $\bar{\theta}_{\mu\nu}(s)$ and $2\pi n_{\mu\nu}(s)$ correspond to the regular field strength and the singular Dirac string part, respectively. Since the Abelian Bianchi identity is broken, the monopole current $k_\mu(s)$ appears on the dual link (s, μ) as

$$k_\mu(s) \equiv (1/4\pi) \varepsilon_{\mu\nu\alpha\beta} \partial_\nu \bar{\theta}_{\alpha\beta}(s + \hat{\mu}) = -\partial_\nu \tilde{n}_{\mu\nu}(s), \quad (6.4)$$

where $\tilde{n}_{\mu\nu}(s) \equiv \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}n_{\alpha\beta}(s + \hat{\mu})$. The current-conservation law $\partial'_\mu k_\mu(*s) = 0$ leads to a closed monopole loop in \mathbf{R}^4 . Here, ∂'_μ denotes a backward derivative.

Here, the self-dual solution automatically satisfies the MA gauge local condition $(\partial_\mu \mp A_\mu^3)A_\mu^\pm = 0$, which gives a stationary point of the functional R_{ch} . For the case of a single instanton on the lattice, the monopole loop is localized around the center of the instanton. However, such a monopole-loop plane cannot be determined, because there is no definite direction in either the single instanton configuration or in the MA gauge condition, due to 4-dim rotational invariance. In this respect, the external field plays an important role in the determination of the monopole trajectory formed by the instanton. Due to the homogeneous external field, the $O(4)$ rotational symmetry is reduced to $O(2) \times O(2)$, which specifies the 2-dim plane of the external field, $F_{12} \neq 0$. In fact, there appears an asymmetry between the xy plane and the zt plane for the $F_{12} \neq 0$ case.

We now apply the MA gauge fixing to the superposed gauge field of the external field and the stabilized instanton that was obtained in the previous section and extract monopole currents. Then, there appears a local correlation between the instanton and monopole. In an external field with $F_{12} > 0$, a monopole loop appears around the instanton configuration in the zt plane, including its center, as shown in Fig. 6. In other words, pair creation of a monopole and an anti-monopole occurs at a time t along the external field $H_3 \neq 0$, and after some time interval pair annihilation occurs, which seems to partially screen the external magnetic field. Figure 7 displays the dependence of the closed area of the monopole loop on the strength of the external field. The external field plays a significant role in the promotion of large monopole-loops.

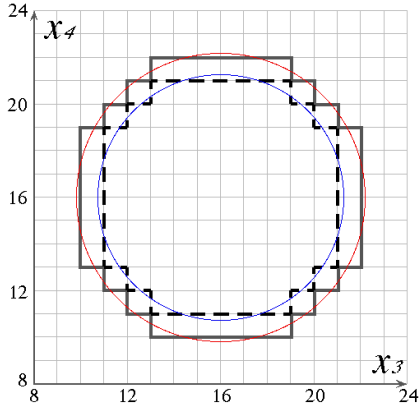


Fig. 6. The monopole loop around an instanton located at the lattice center in an external field after the action minimization procedure has been applied. Here, we use the MA gauge. The dashed and solid curves denote monopole loops for $F_{12}/4\pi = 1$ and $\sqrt{3} \text{ fm}^{-2}$, corresponding to $\langle G_{\mu\nu}^a G_{\mu\nu}^a \rangle / 32\pi^2 = 1$ and 3 fm^{-4} , respectively.

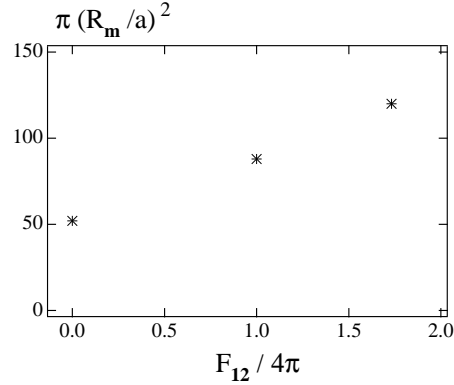


Fig. 7. The dependence of the minimal area enclosed by the monopole loop on the strength of the external field $F_{12}/4\pi$. Here, R_m is the radius of the monopole loop.

§7. Summary and conclusion

In this paper, we have investigated a deformation mechanism of the instanton configuration in the presence of a homogeneous external field. An isolated single instanton has global color orientations O^{ab} as collective coordinates and gauge degrees of freedom $\Omega(x)$. However, external fields lead to the dependence of the total Yang-Mills action on these variables and the breaking of the $O(4)$ symmetry of the instanton solution. In this situation, the 't Hooft ansatz is not a solution of the Yang-Mills field equation and is unstable in the variational space of the instanton configuration, whose degrees of freedom are the global color orientation O^{ab} , the anisotropic scale λ , and the local gauge-like function $\Omega(x)$. In this way, we have found the stable instanton solution that minimizes the total Yang-Mills action starting from a total gauge field that is a superposition of the 't Hooft ansatz in the singular gauge with an external field.

We have investigated the case of a homogeneous external field $F_{12} > 0$ for simplicity and determined O^{ab} , θ and $\Omega(x)$ so as to minimize the total action. Due to the external field, the $O(4)$ symmetry is reduced to two 2-dimensional rotational symmetries $O(2) \times O(2)$ in the xy and zt planes in the instanton solution. In the space \mathbf{R}^3 at fixed t , we found a quadrupole deformation of this instanton solution. In a magnetic field \vec{H} , a prolate deformation occurs along the direction of \vec{H} . The magnitude of the deformation is found to increase with the instanton size and the strength of background fields. Since the QCD vacuum is composed of many instantons and anti-instantons, instanton configurations seem to lose the $O(4)$ symmetry and become deformed by such background fields.

We have considered further the effect of external fields on the local correlation between instantons and monopoles in the MA gauge. Both for the single instanton and MA gauge conditions, there is no definite direction in \mathbf{R}^4 , and therefore in the absence of a background field, the direction of a monopole loop is “fragile”. A homogeneous external field breaks the 4-dimensional rotational invariance on the instanton configuration, and as a result, a monopole trajectory around the instanton is formed with a direction corresponding to the external field. A monopole and anti-monopole pair is created at a time t along the z direction of the external field $H_3 \neq 0$ and after some time interval pair annihilation occurs in a zt plane. In fact, such a monopole loop seems to partially screen the external field. The closed minimal area of the monopole-loop generated by a fixed size instanton is found to increase with the strength of the external field.

Acknowledgements

We would like to thank Dr. H. Toki for his useful comments and discussions. We have carried out all numerical simulations reported in this paper on a NEC SX5 at RCNP.

Appendix

We must maintain the geometrical symmetry and the ordering symmetry for the product of U_μ^I and U_μ^{ex} in the minimization procedure. To maintain the ordering symmetry for the product of U^I and U^{ex} , we have to consider both of the ordering choices $U^R = U^{ex}U^I$ and $U^L = U^IU^{ex}$, for each direction μ and ν . As shown in Fig. 8, we therefore consider four types of plaquettes (1),(2),(3) and (4), which correspond to (RR), (LL), (RL) and (LR), respectively. Here, we use the clover-type plaquettes because of the geometrical symmetry. The cross-checked plaquette in each clover is independent of the local gauge transformation of $\Omega(s_0)$ and does not contribute to $\delta S_{YM}(s_0; \Omega(s_0))$. We list 12 components of $\hat{L}_{\mu\nu}^\alpha$ and $\bar{L}_{\mu\nu}^\alpha$ in Eq. (4-10) for the construction of δS_{YM} as follows.

The RR part is written

$$\hat{L}_{\mu\nu}^1(s_0) = U_\nu^{ex\dagger}(s_0)U_\mu^{ex}(s_0),$$

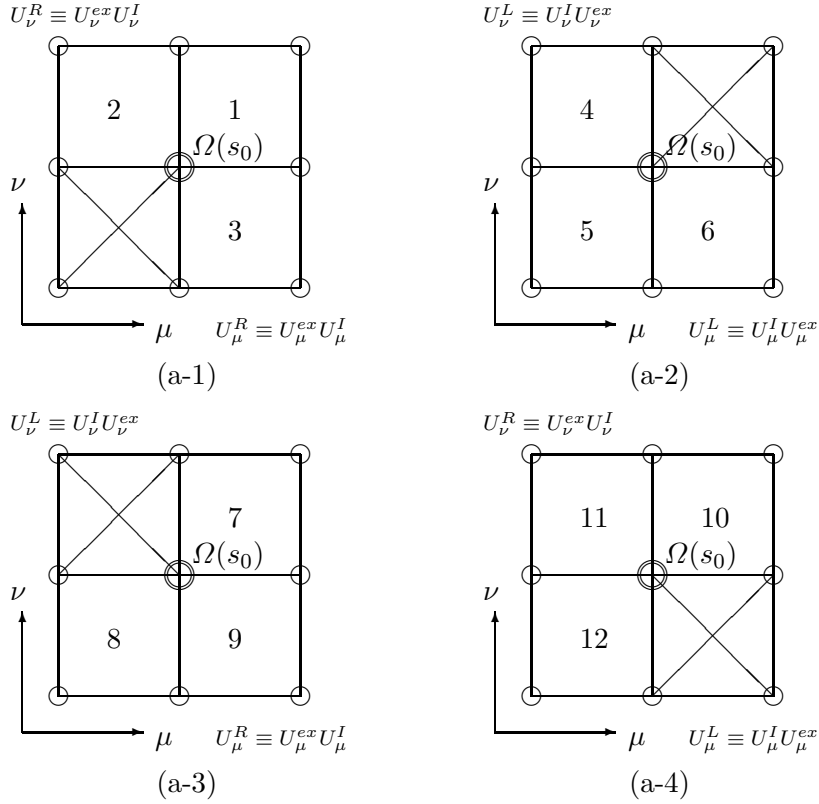


Fig. 8. Wilson plaquette around a site s_0 . Here, we use the gauge transformation $\Omega(s_0)$ of s_0 . To maintain the ordering symmetry for the product of U_μ^I and U_μ^{ex} , we have to consider both of the ordering choices $U^R = U^{ex}U^I$ and $U^L = U^IU^{ex}$ for each direction μ and ν . Therefore, we consider four types of clover plaquettes, (1),(2),(3) and (4), which correspond to (RR), (LL), (RL) and (LR), respectively. The cross-checked plaquette in each clover is independent of the local gauge transformation of $\Omega(s_0)$, and thus does not contribute to $\delta S_{YM}(s_0; \Omega(s_0))$.

$$\begin{aligned}
\bar{L}_{\mu\nu}^1(s_0) &= U_\mu^I(s_0)U_\nu^R(s_0 + \hat{\mu})U_\mu^{R\dagger}(s_0 + \hat{\nu})U_\nu^{I\dagger}(s_0), \\
\hat{L}_{\mu\nu}^2(s_0) &= U_\nu^{ex}(s_0), \\
\bar{L}_{\mu\nu}^2(s_0) &= U_\nu^I(s_0)U_\mu^{R\dagger}(s_0 - \hat{\mu} + \hat{\nu})U_\nu^{R\dagger}(s_0 - \hat{\mu})U_\mu^R(s_0 - \hat{\mu}), \\
\hat{L}_{\mu\nu}^3(s_0) &= U_\mu^{ex\dagger}(s_0), \\
\bar{L}_{\mu\nu}^3(s_0) &= U_\nu^{R\dagger}(s_0 - \hat{\nu})U_\mu^R(s_0 - \hat{\nu})U_\nu^R(s_0 + \hat{\mu} - \hat{\nu})U_\mu^{I\dagger}(s_0). \tag{A.1}
\end{aligned}$$

The LL part is written as

$$\begin{aligned}
\hat{L}_{\mu\nu}^4(s_0) &= U_\mu^{ex}(s_0 - \hat{\mu}), \\
\bar{L}_{\mu\nu}^4(s_0) &= U_\nu^L(s_0)U_\mu^{L\dagger}(s_0 - \hat{\mu} + \hat{\nu})U_\nu^{L\dagger}(s_0 - \hat{\mu})U_\mu^I(s_0 - \hat{\mu}), \\
\hat{L}_{\mu\nu}^5(s_0) &= U_\nu^{ex}(s_0 - \hat{\nu})U_\mu^{ex\dagger}(s_0 - \hat{\mu}), \\
\bar{L}_{\mu\nu}^5(s_0) &= U_\mu^{I\dagger}(s_0 - \hat{\mu})U_\nu^{L\dagger}(s_0 - \hat{\mu} - \hat{\nu})U_\mu^L(s_0 - \hat{\mu} - \hat{\nu})U_\nu^I(s_0 - \hat{\nu}), \\
\hat{L}_{\mu\nu}^6(s_0) &= U_\nu^{ex\dagger}(s_0 - \hat{\nu}), \\
\bar{L}_{\mu\nu}^6(s_0) &= U_\nu^{I\dagger}(s_0 - \hat{\nu})U_\mu^L(s_0 - \hat{\nu})U_\nu^L(s_0 + \hat{\mu} - \hat{\nu})U_\mu^{L\dagger}(s_0). \tag{A.2}
\end{aligned}$$

The RL part is written as

$$\begin{aligned}
\hat{L}_{\mu\nu}^7(s_0) &= U_\mu^{ex}(s_0), \\
\bar{L}_{\mu\nu}^7(s_0) &= U_\mu^I(s_0)U_\nu^L(s_0 + \hat{\mu})U_\mu^{R\dagger}(s_0 + \hat{\nu})U_\nu^{L\dagger}(s_0), \\
\hat{L}_{\mu\nu}^8(s_0) &= U_\nu^{ex}(s_0 - \hat{\nu}), \\
\bar{L}_{\mu\nu}^8(s_0) &= U_\mu^{R\dagger}(s_0 - \hat{\mu})U_\nu^{L\dagger}(s_0 - \hat{\mu} - \hat{\nu})U_\mu^R(s_0 - \hat{\mu} - \hat{\nu})U_\nu^I(s_0 - \hat{\nu}), \\
\hat{L}_{\mu\nu}^9(s_0) &= U_\mu^{ex\dagger}(s_0)U_\nu^{ex\dagger}(s_0 - \hat{\nu}), \\
\bar{L}_{\mu\nu}^9(s_0) &= U_\nu^{I\dagger}(s_0 - \hat{\nu})U_\mu^R(s_0 - \hat{\nu})U_\nu^L(s_0 + \hat{\mu} - \hat{\nu})U_\mu^{I\dagger}(s_0). \tag{A.3}
\end{aligned}$$

The LR part is written as

$$\begin{aligned}
\hat{L}_{\mu\nu}^{10}(s_0) &= U_\nu^{ex\dagger}(s_0), \\
\bar{L}_{\mu\nu}^{10}(s_0) &= U_\mu^L(s_0)U_\nu^R(s_0 + \hat{\mu})U_\mu^{L\dagger}(s_0 + \hat{\nu})U_\nu^{I\dagger}(s_0), \\
\hat{L}_{\mu\nu}^{11}(s_0) &= U_\mu^{ex}(s_0 - \hat{\mu})U_\nu^{ex}(s_0), \\
\bar{L}_{\mu\nu}^{11}(s_0) &= U_\nu^I(s_0)U_\mu^{L\dagger}(s_0 - \hat{\mu} + \hat{\nu})U_\nu^{R\dagger}(s_0 - \hat{\mu})U_\mu^I(s_0 - \hat{\mu}), \\
\hat{L}_{\mu\nu}^{12}(s_0) &= U_\mu^{ex\dagger}(s_0 - \hat{\mu}), \\
\bar{L}_{\mu\nu}^{12}(s_0) &= U_\mu^{I\dagger}(s_0 - \hat{\mu})U_\nu^{R\dagger}(s_0 - \hat{\mu} - \hat{\nu})U_\mu^L(s_0 - \hat{\mu} - \hat{\nu})U_\nu^R(s_0 - \hat{\nu}). \tag{A.4}
\end{aligned}$$

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